Building lampposts in moduli spaces

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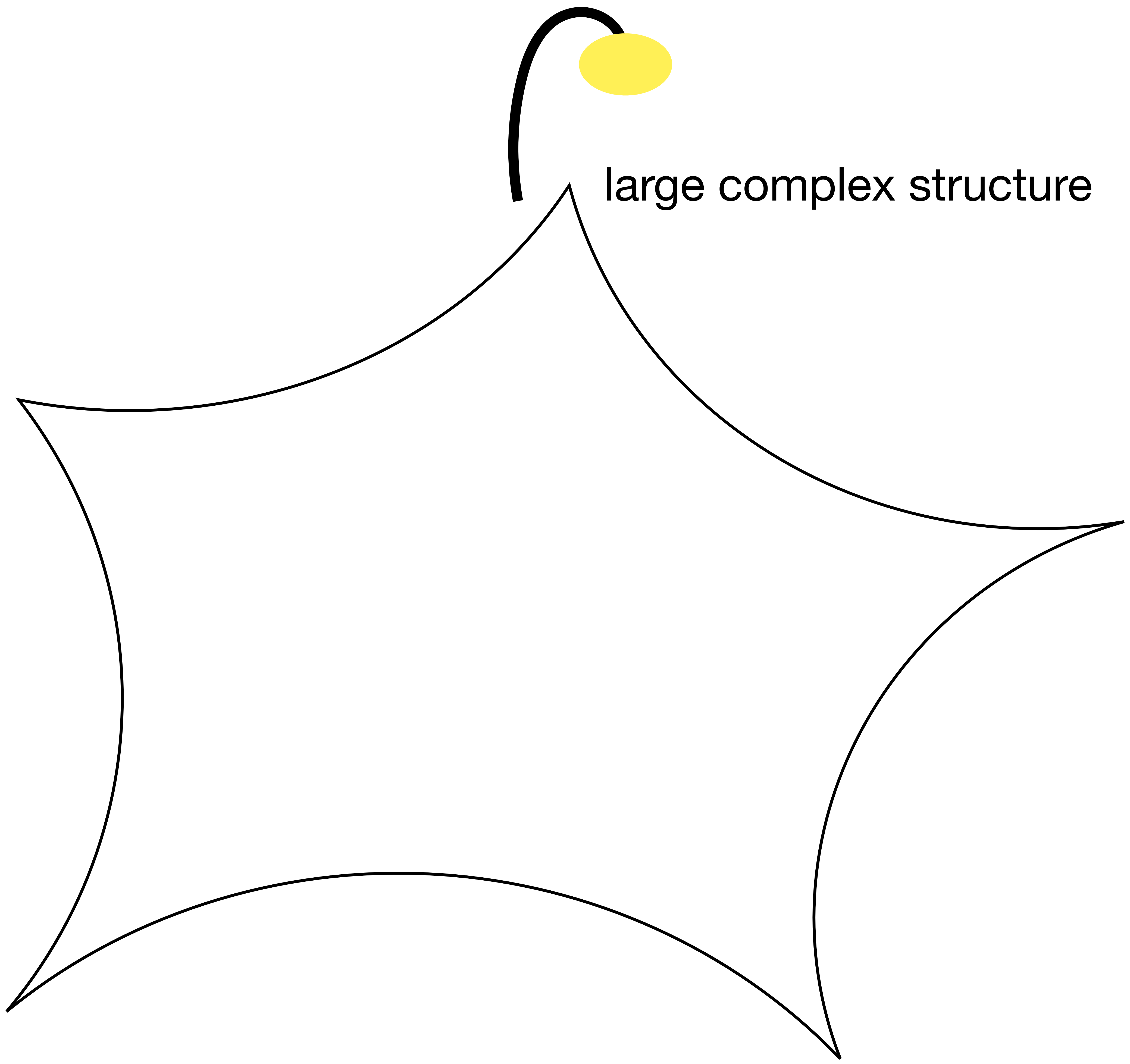
Based on: work in progress, with **Brice Bastian**, **Thomas Grimm** and **Lorenz Schlechter**
(2105.02232, with **Brice Bastian** and **Thomas Grimm**)

String phenomenology 2022
Liverpool

Setting the stage

Arena:

Complex structure moduli space of Calabi-Yau manifolds



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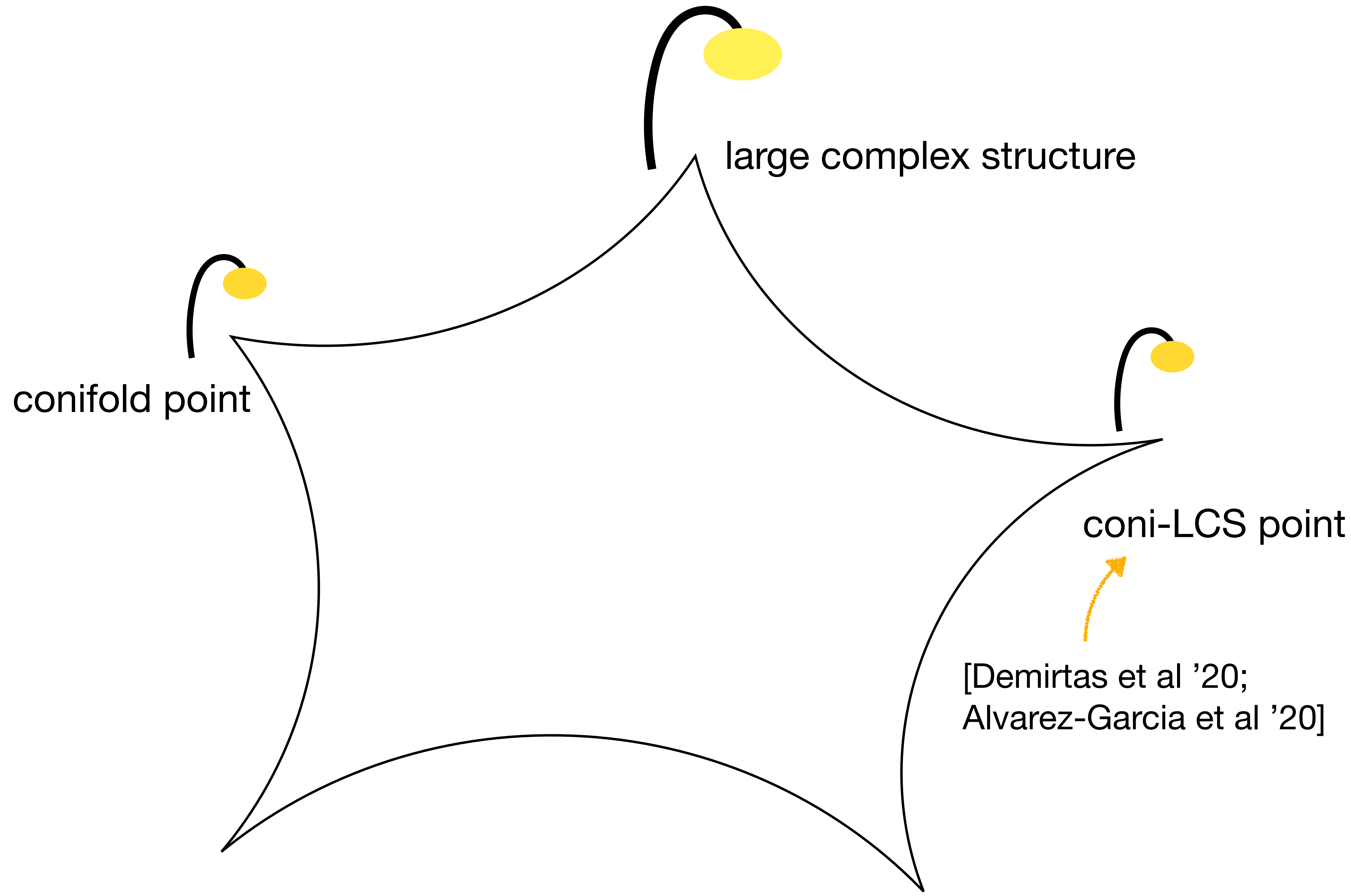


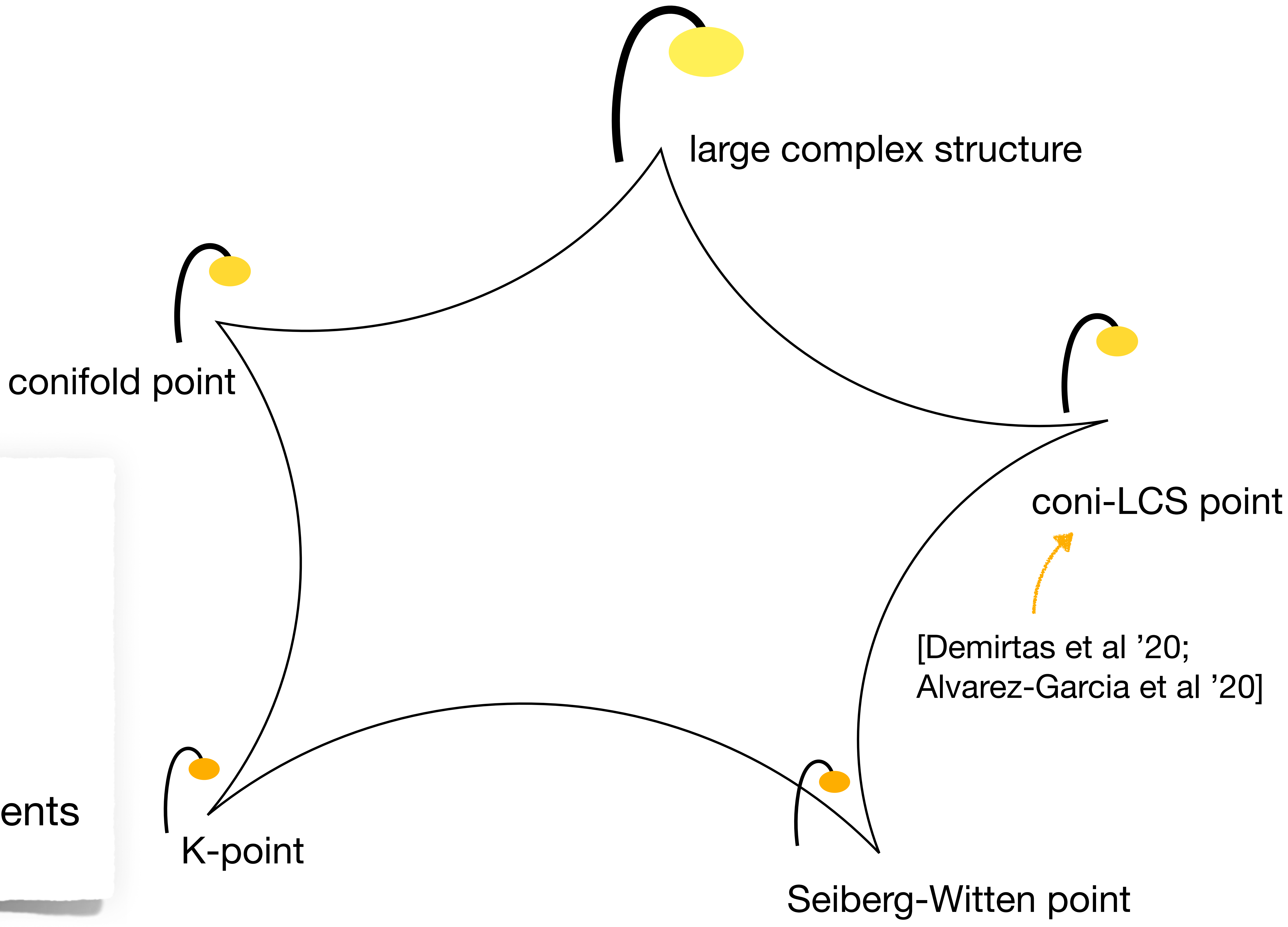
Setting the stage

Arena:

Complex structure moduli space of Calabi-Yau manifolds

Goal:

- Explore boundaries away from LCS
  - Write down prepotentials/periods
  - Characterize model-dependent coefficients



$\Longrightarrow$ **asymptotic Hodge theory** shines a light on all boundaries [Grimm, Palti, Valenzuela ’18; …]

Physical couplings and periods

Physical couplings in string compactifications:

- Kähler potential of Type IIB CY compactifications $K = -\log i \int_{Y_3} \Omega \wedge \bar{\Omega}$
- Flux superpotential of Type IIB CY orientifolds $W = \int_{Y_3} G_3 \wedge \Omega$

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Periods:

Physical couplings determined by **period integrals** of the (3,0)-form:

$$\Pi^I(z) = \int_{\Gamma_I} \Omega(z)$$

$\Longrightarrow$ complicated transcendental functions (can be computed in **examples** with e.g. Picard-Fuchs methods: [Hosono, Klemm, Theisen, Yau ’94; …])

Large complex structure point

Periods near LCS: $$\Pi = \begin{pmatrix} 1 \\ t \\ \frac{1}{6}\kappa_{111}t^3 + b_1 t + \frac{i\chi\zeta(3)}{8\pi^3} \\ -\frac{1}{2}\kappa_{111}t^2 + b_1 \end{pmatrix} + \mathcal{O}(e^{2\pi i t})$$

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Wishlist for periods near other boundaries

- Geometric interpretation of **leading coefficients** in periods (topological data of mirror CY)
- **Natural coordinate** around singularity (mirror map)
- Understanding for **exponential corrections** (worldsheet instantons, GV invariants)
- (More pragmatic: expression for the **prepotential**)

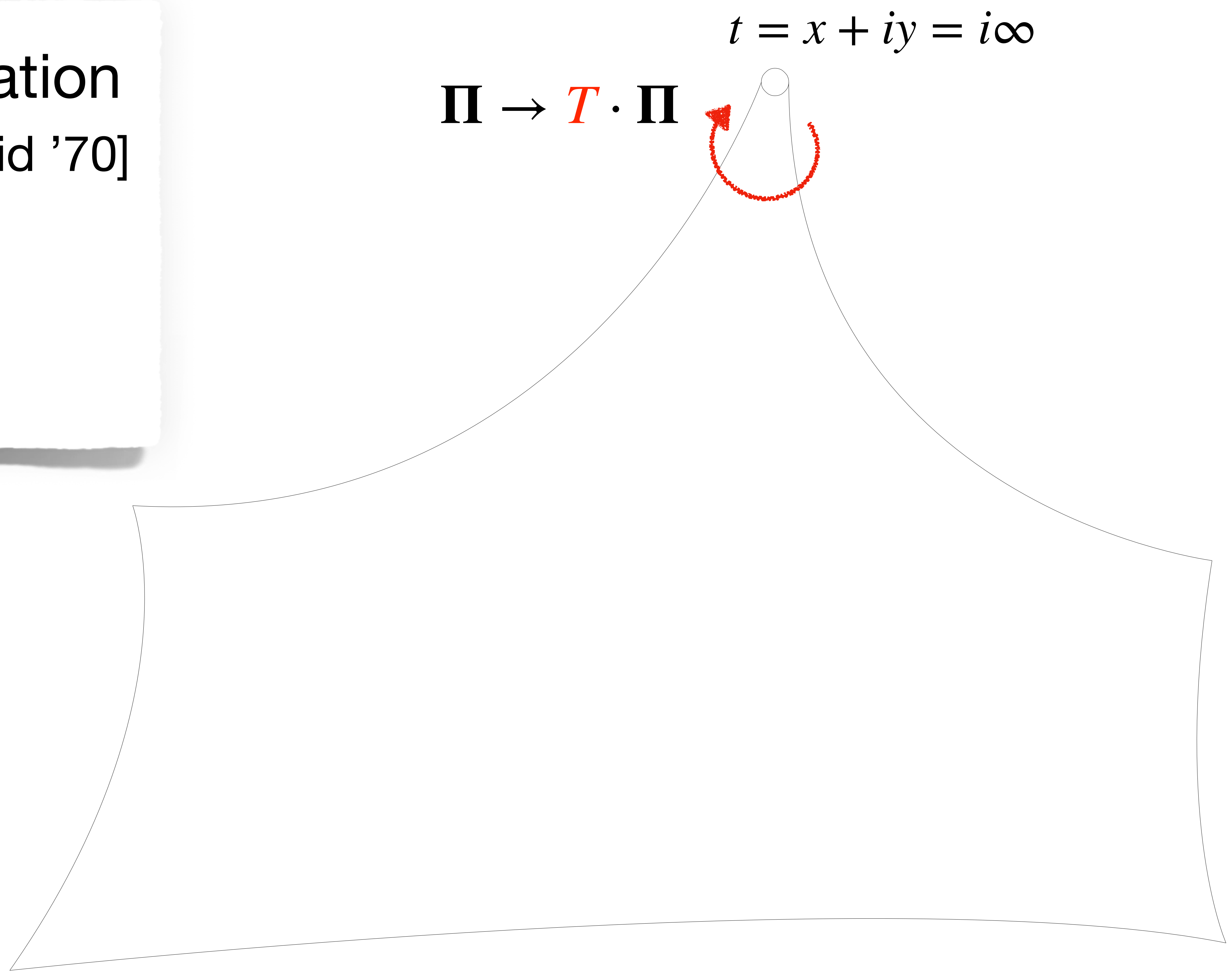
Asymptotic behavior of periods

Near-boundary behavior: nilpotent orbit approximation

[Schmid '70]

$$\mathbf{\Pi}(t) = e^{tN}\left(\mathbf{a}_0 + \sum_{r>0} e^{2\pi i r t}\,\mathbf{a}_r\right)$$

"perturbative terms" "instanton corrections"



- Nilpotent log-monodromy matrices $N = \log T$

  ($N^4 = 0$ for CY threefolds)

- Exponential corrections $\mathbf{a}_r$ are **essential** near boundaries away from LCS lamppost [Bastian, Grimm, DH '21a]

fits nicely with [Palti, Vafa, Weigand '20]

[Cecotti, '20a]

Asymptotic period models

Construction of asymptotic periods [Bastian, Grimm, DH ‘21a]

- General models for all possible one- and two-moduli boundaries
- Includes essential exponential corrections

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Still to do: [Bastian, Grimm, DH, Schlechter]

- Periods in **integral** basis (important for flux quantization)

  $\Longrightarrow$ extension data methods [Green, Griffiths, Kerr ’08]

- Understand **model-dependent coefficients**

  $\Longrightarrow$ match with geometrical examples

Boundaries and singular geometries

Boundaries:

Calabi-Yau threefold degenerates at the boundaries

Asymptotic regimes:

Topological & arithmetic properties of singular geometry determine **leading period coefficients**

[Bastian, Grimm, DH, Schlechter],
see also [Bönisch, Klemm, Scheidegger, Zagier ’22]

Geometrical input

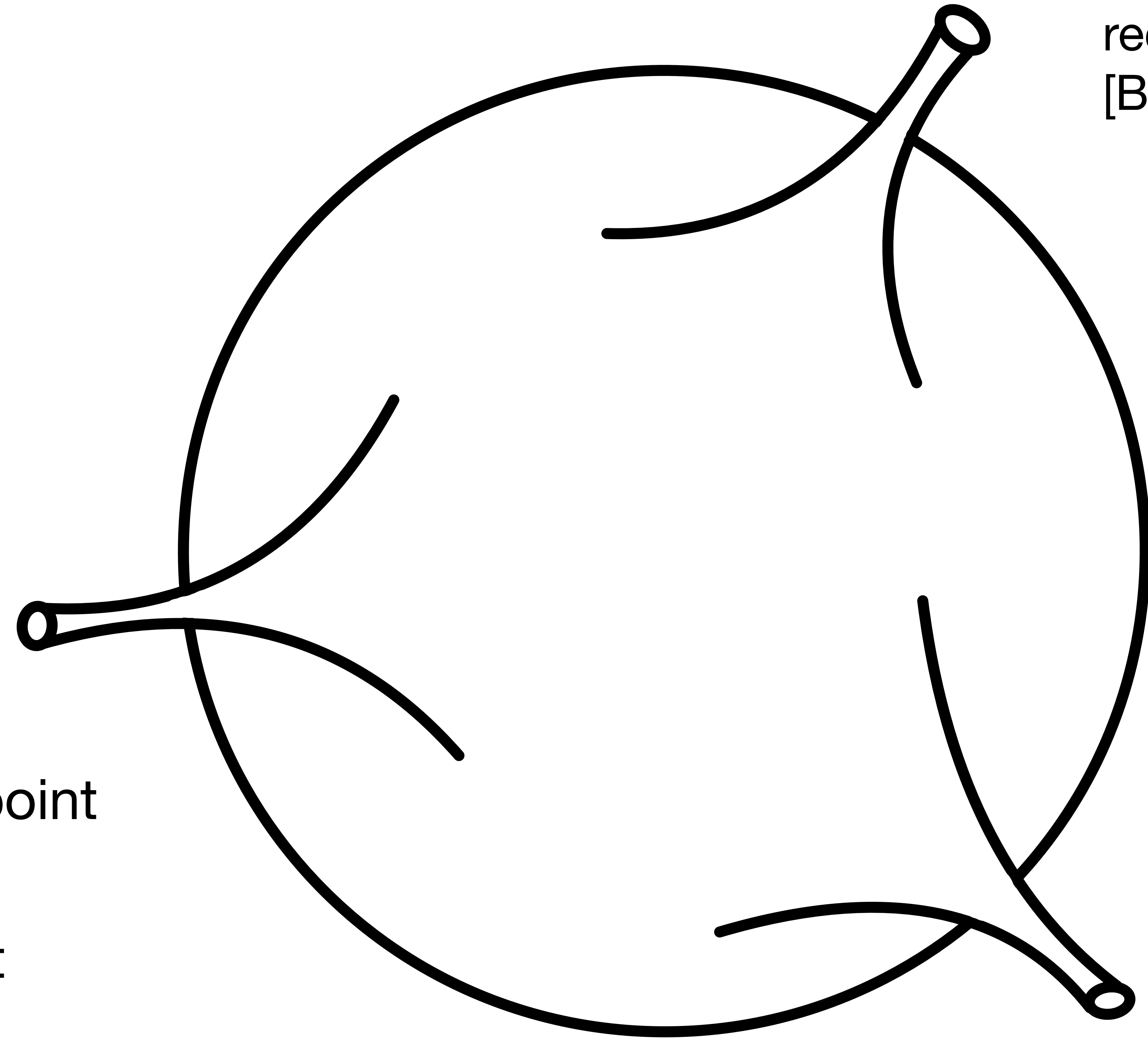
14 hypergeometric examples see e.g. [Van Straten, ’18]

Complex structure moduli space: $\mathbb{P}^1 \backslash \{0,1,\infty\}$

$z = 1$

- conifold point

recently in
[Bönisch, Klemm, Scheidegger, Zagier ’22]



$z = 0$

- LG point
- conifold point
- K-point
- LCS point

$z = \infty$

- large complex structure (LCS)

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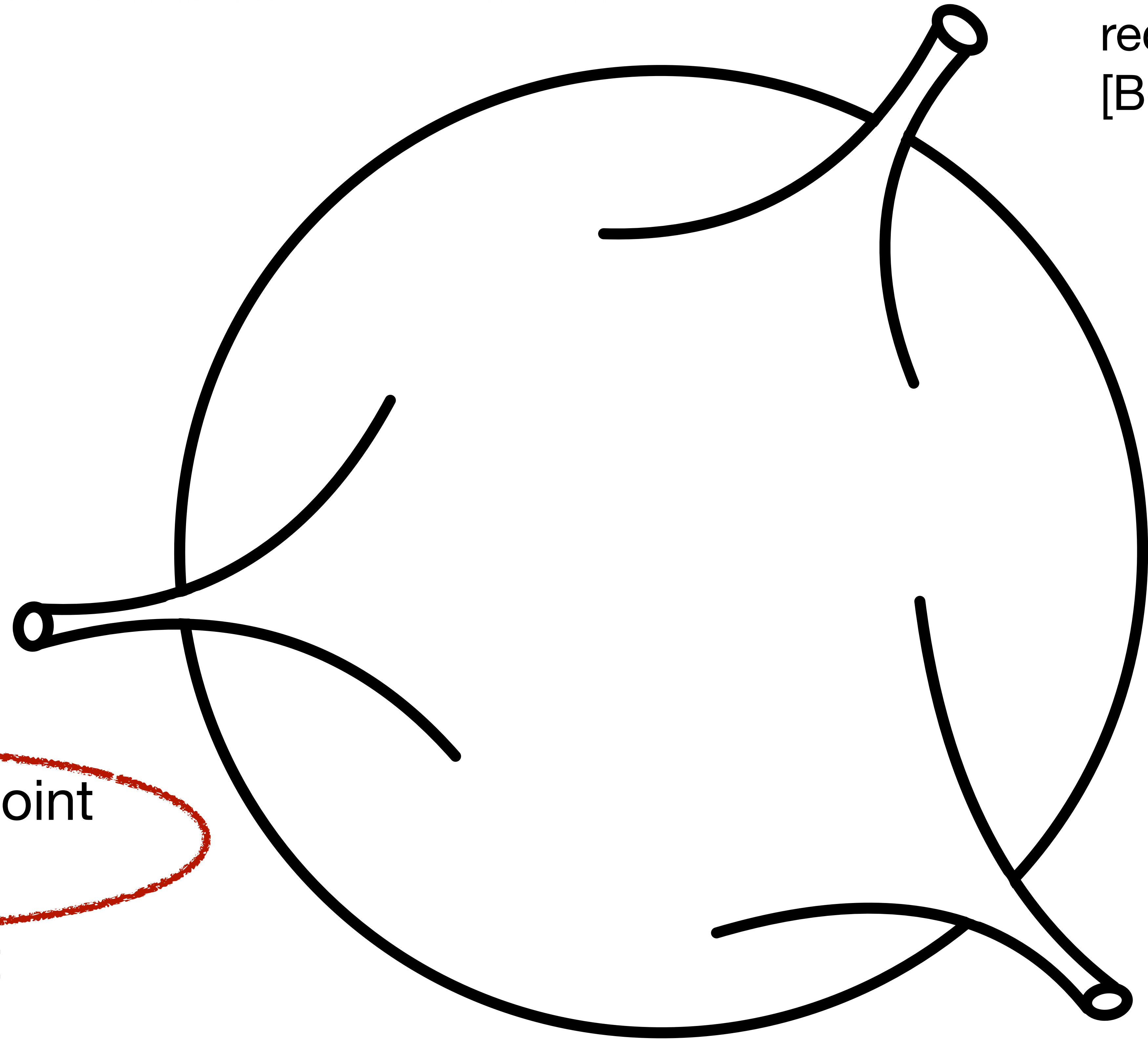


Our focus

$z = 0$

- LG point
- conifold point
- K-point
- LCS point

$z = \infty$

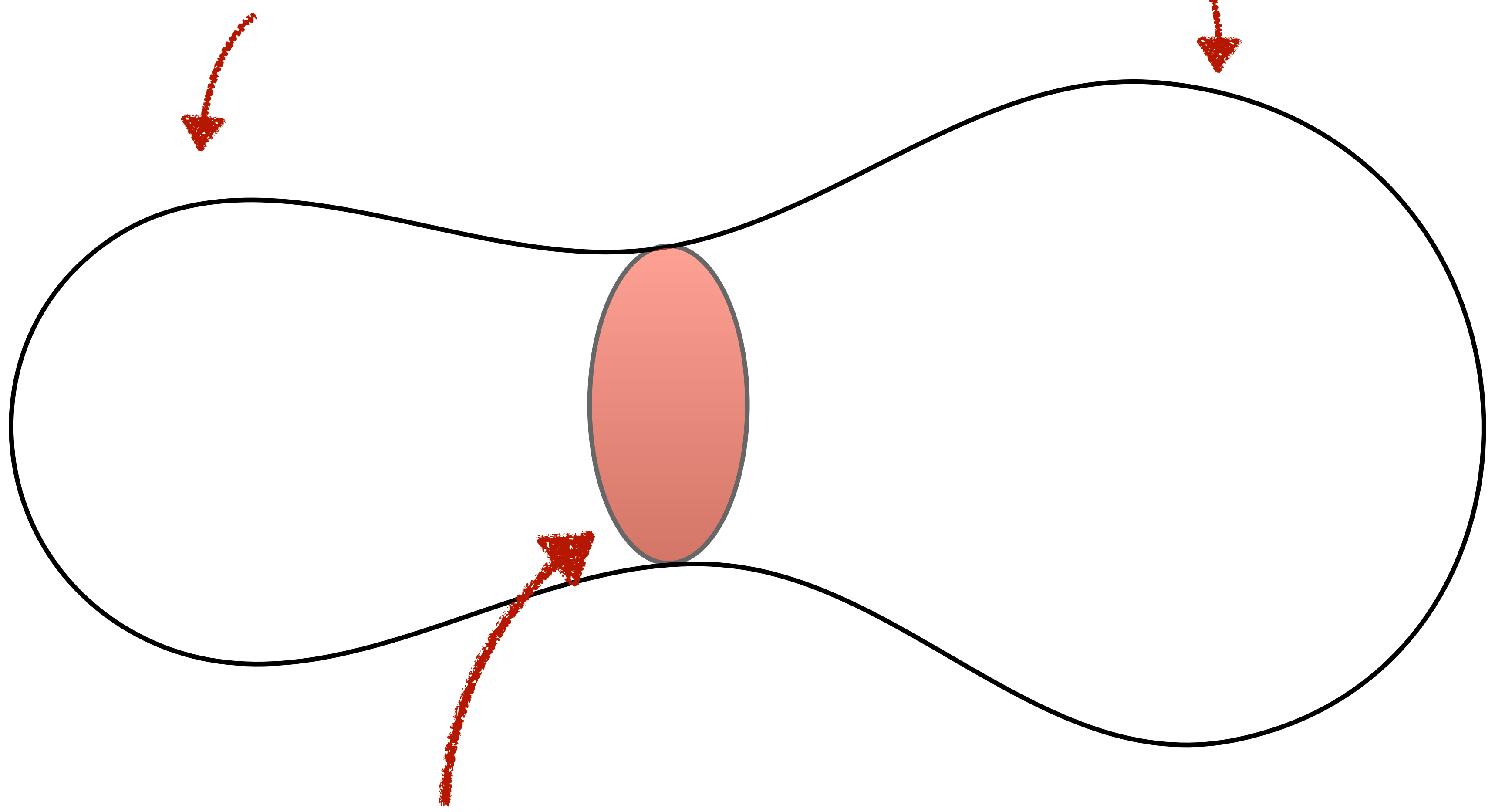
- large complex structure (LCS)

(Patches related by $z = e^{2\pi i t}$)

K-point: geometry (also: Tyurin degeneration, II_0 singularity)

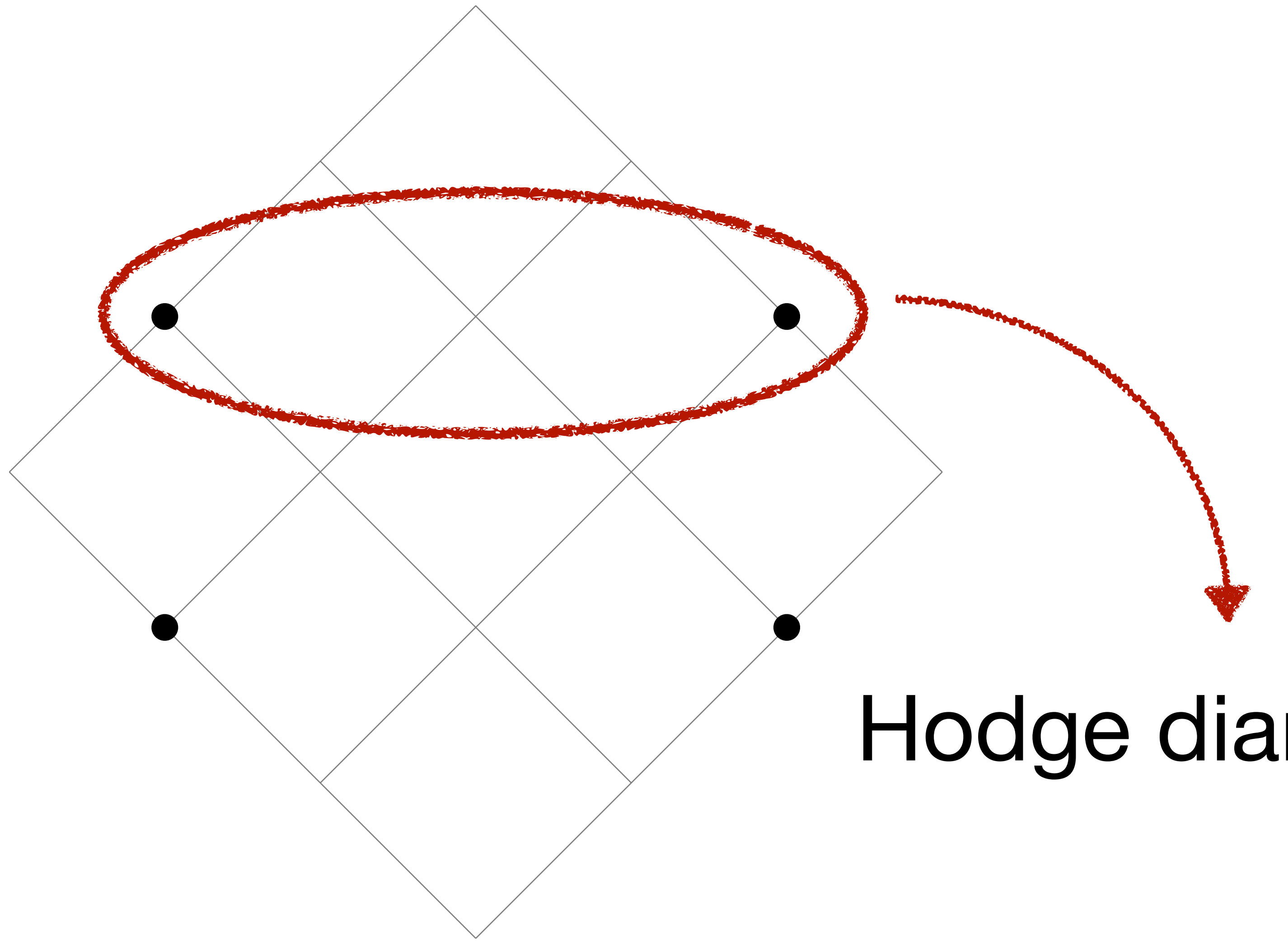
Geometry at the limit:

(quasi-)Fano threefolds



Rigid K3 surface (no complex structure moduli)

Limiting mixed Hodge structure:



Hodge diamond of rigid K3

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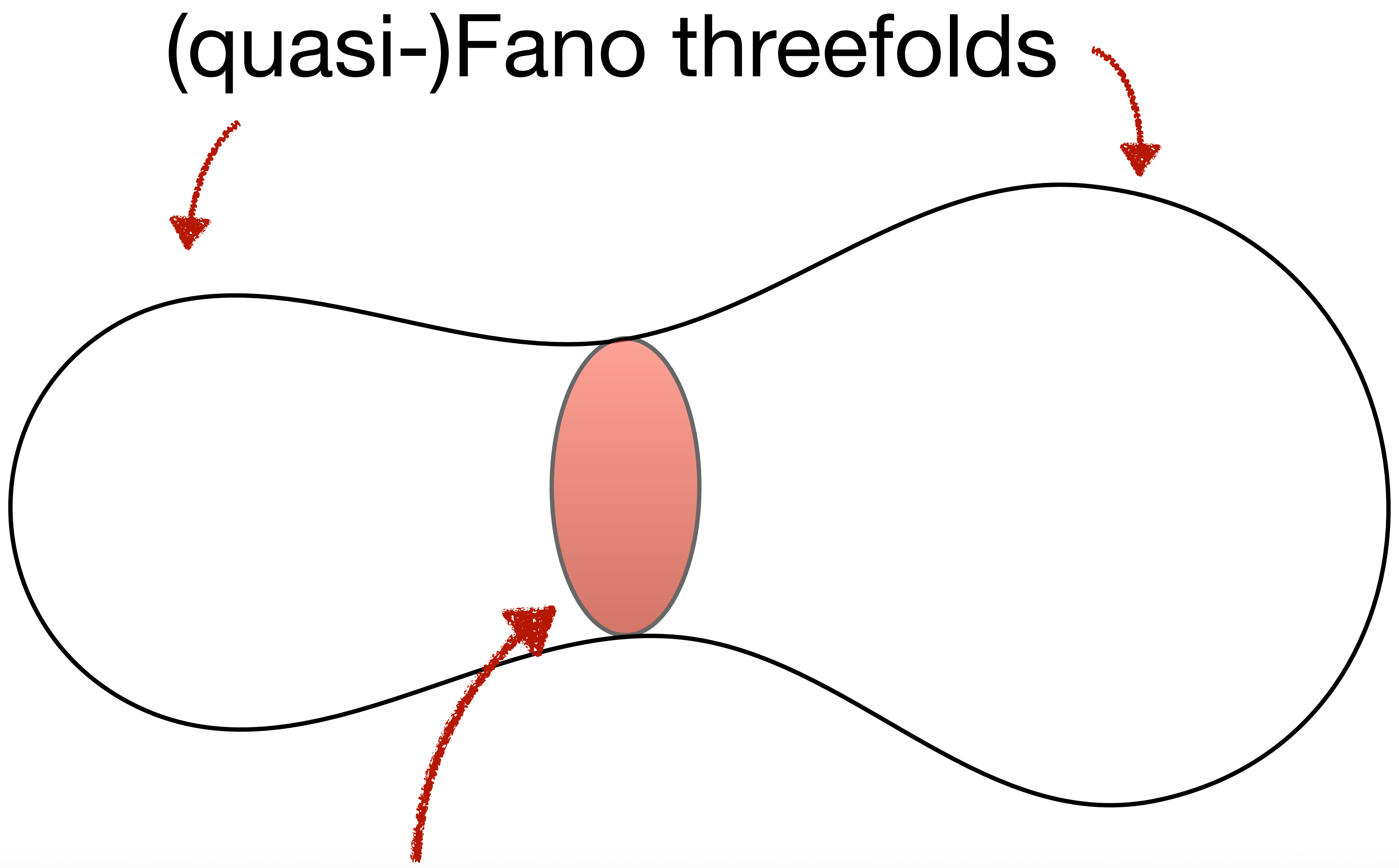

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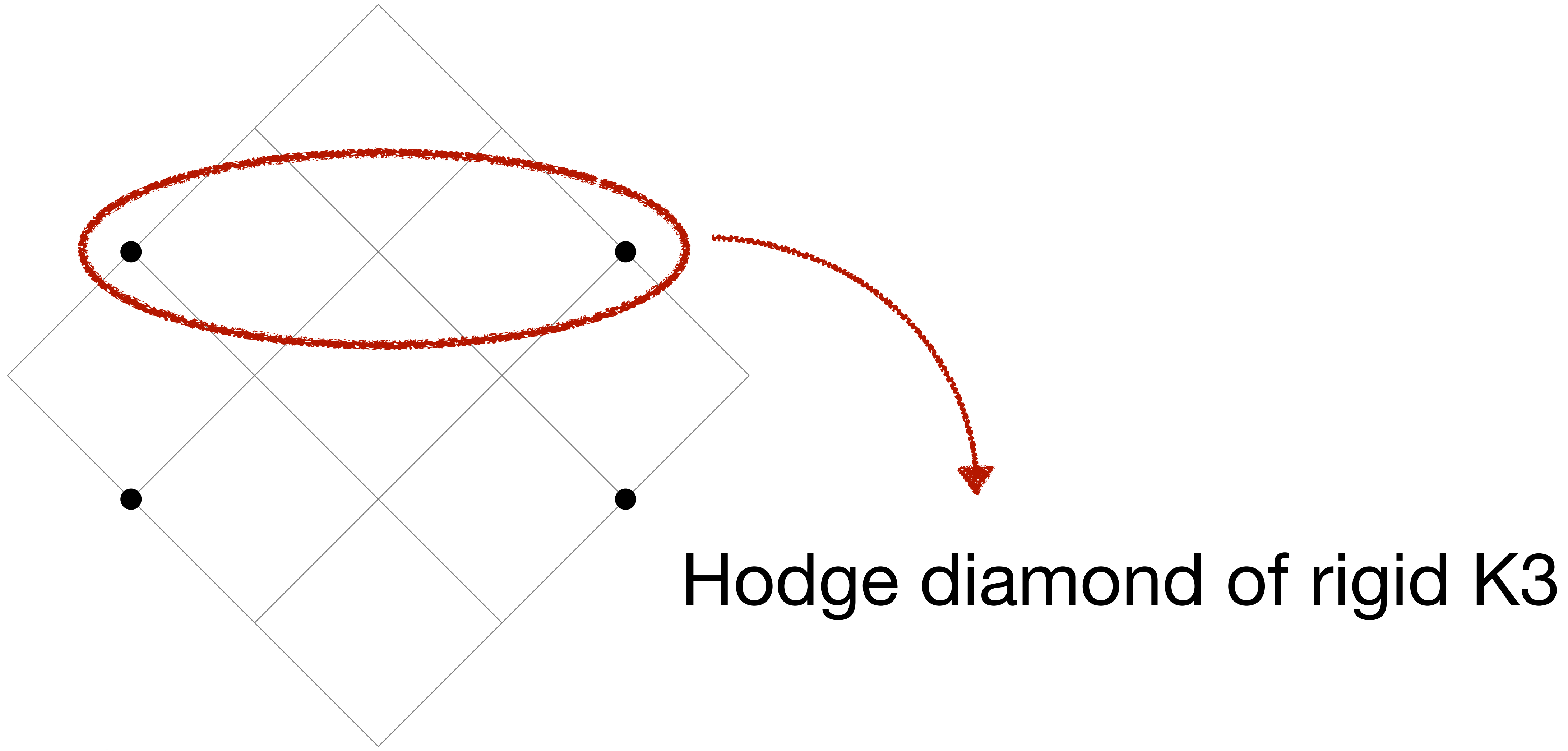

Rigid K3 surface (no complex structure moduli)

Characterized by intersection form on $H^{2,0} \oplus H^{0,2} \subset H^2(\mathrm{K3})$

$$\int_{K3} \cdot \wedge \cdot = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \text{ with } d = 4ac - b^2 > 0$$

K-point: periods

[Bastian, Grimm, DH, Schlechter]

Asymptotic periods:

$$\Pi = \begin{pmatrix} 1 \\ z - \tau \\ 2F - (z-\tau)\partial_z F \\ \partial_z F \end{pmatrix}$$

prepotential $F = A + z(\sqrt{d}\frac{\log z}{2\pi} + B) + z^2(c\sqrt{d}\frac{\log z}{2\pi} + C) + \ldots$

$$\tau = \frac{-b \pm i\sqrt{d}}{2c}$$

log-monodromy matrix $N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a & b & 0 & 0 \\ b & 2c & 0 & 0 \end{pmatrix}$

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transcendental

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pairing of K3

Remarks:

- Coefficient $\sqrt{d}$ characterizes deviation away from rational numbers

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Remarks:

- Coefficient $\sqrt{d}$ characterizes deviation away from rational numbers
- Explicit examples for small W_0 vacua proposed in [Bastian, Grimm, DH '21b]

(with $\mathcal{O}(1)$ moduli masses)

Modularity and period coefficients

What about transcendental numbers in periods?

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$\Longrightarrow$ encoded in **modular forms** associated to singular geometry:

Calabi-Yau manifolds at boundary $\longleftrightarrow$ Modular forms $f(q) = \sum_n c_n q^n$

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Coefficients computed from modular form as **L-values**:

[Bönisch, Klemm, Scheidegger, Zagier '22]
[Alvarez-Garcia, Blumenhagen, Brinkmann, Schlechter '20]
[Bastian, Grimm, DH, Schlechter]

$$L(f,x) = \sum_n \frac{c_n}{n^x}$$

(similar to $\zeta(3)$ in LCS periods)

Related work: attractor points & supersymmetric flux vacua [Candelas, de la Ossa, Elmi, Van Straten '19; Kachru, Nally, Yang '20 & '21]

also talks by Fabian Ruehle, Liam McAllister, Naomi Gendler

Conifold point [Bastian, Grimm, DH, Schlechter]

also [Bönisch, Klemm, Scheidegger, Zagier ’22;
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Conifold periods: $\Pi = \begin{pmatrix} 1 \\ z \\ 2\,F - z\,\partial_z F \\ \partial_z F \end{pmatrix}$

prepotential $F_c = \frac{\tau}{2} + Az + z^2(k\frac{\log z}{4\pi i} + B) + \ldots$

$\mathbb{Q}(\tau)$ positive integer

L-value parameter

$\mathrm{Re}(\tau) = 0, \frac{1}{2}$

$\mathrm{Im}(\tau) = r\frac{L(f,2)}{L(f,1)} > 0 \quad (r \in \mathbb{Q}^*)$

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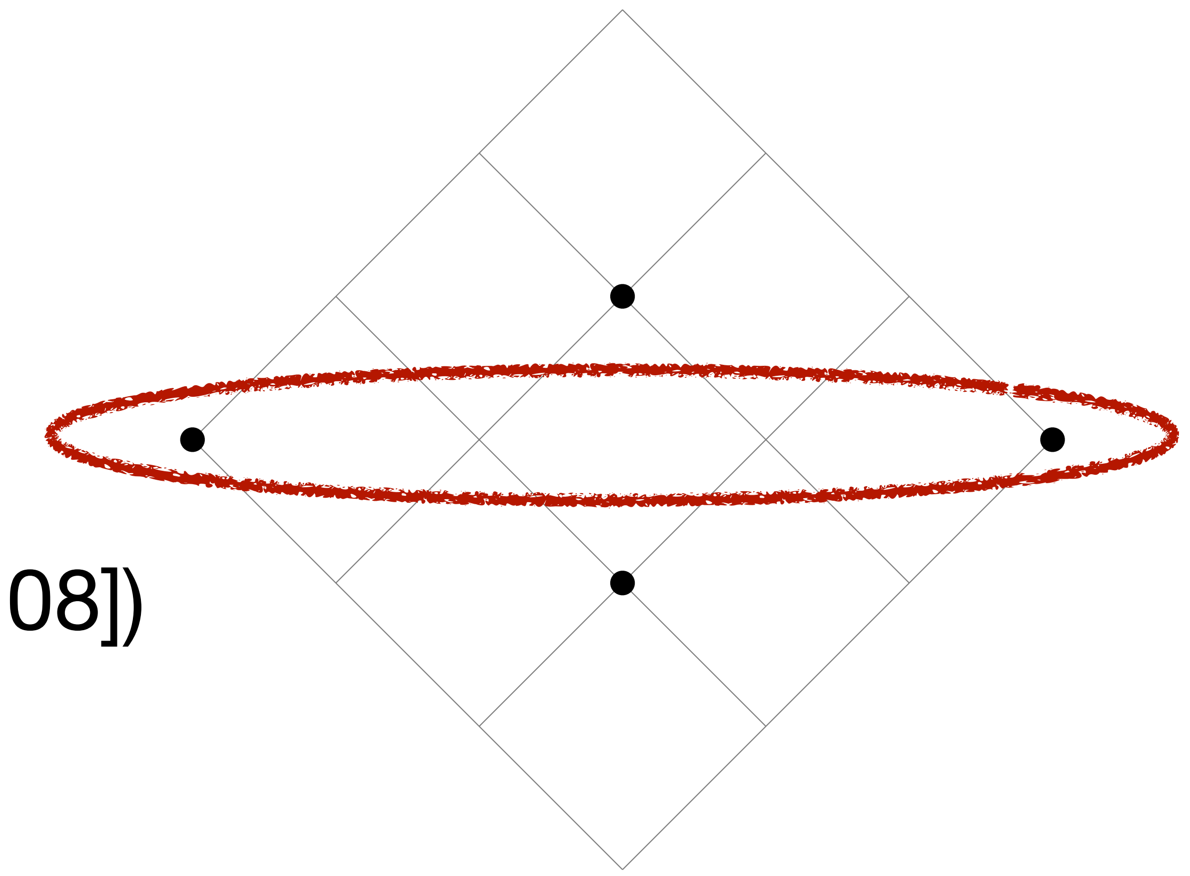
Where does this modularity come from?

$\Longrightarrow$ rigid conifold in LMHS:



(Rigid CYs are modular by [Gouvea, Yui ’08])

Conclusions & outlook

Conclusions:

- **Asymptotic Hodge theory** illuminates all boundaries in moduli space
- **Singular geometry** at the boundary encodes **leading coefficients** in periods

  $\Longrightarrow$ **topological** and **arithmetic** numbers can be extracted from databases
- **Rational** coefficients for flux quantization in model building

Outlook:

- Promising to extend to **multi-moduli** setups
- Interpretation for coefficients of **exponential corrections**

Thank you for your attention!